



TITLE:

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The family of $K3$ surfaces with a transcendental lattice $U(2)^2 \times \langle -2 \rangle^4$ for a general member

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0 Introduction

Let us consider the lattice $P = D_4^3 \oplus \langle -2 \rangle \oplus \langle 2 \rangle$ of rank 14. A $K3$ surface S is called of type P when it satisfies $P \subset \text{Pic}(S)$, where $\text{Pic}(S)$ stands for the Picard lattice of S . In this article we show the outline of the study on the period map for the family \mathcal{F} of $K3$ surfaces of type P .

We work always on the ground field \mathbb{C} . Note that the lattice $H^2(S, \mathbb{Z})$ is always isomorphic to $L = U^3 \oplus (-E_8)^2$ and $P^\perp \subset L$ is isomorphic to $U(2)^2 \oplus \langle -2 \rangle^4$.

In 1992 K. Matsumoto, T. Sasaki and M. Yoshida [7] studied the period mapping for a family of $K3$ surfaces of type $(3, 6)$, that is the family of double covering surfaces over \mathbb{P}^2 branching along six lines in general position, and Matsumoto [6] gave the description of the inverse mapping in terms of theta constants. It gives the modular map for the 4 dimensional Shimura variety in the Siegel upper half space \mathcal{S}_4 derived from the family of 4 dimensional abelian varieties with generalized complex multiplication by $\sqrt{-1}$ of type $(2, 2)$. So we call it MSY modular mapping. The second author showed an arithmetic application of MSY modular mapping in [11].

In the case of MSY modular map the corresponding $K3$ surface is characterized by the lattice $U(2)^2 \oplus \langle -2 \rangle^2$ and the moduli space is a 4-dimensional type IV domain. We suspect such fruitful results of the MSY map is the consequence of the eventual coincidence of two different bounded symmetric domains D_{IV}^4 and $I_{2,2}$. There are a few (finite) such exceptional coincidences. The highest one is the (analytic) equivalence between D_{IV}^6 and $\mathbf{H}_{II}(4)$ (in terms of Lie algebra $so(2, 6; \mathbb{R}) \cong so(4, \mathbb{F})$, where \mathbb{F} indicates the Hamilton quaternion field) and it contains the above coincidence of MSY case. That is our situation.

The ring structure of the regular functions on the parameter space for \mathcal{F} is given by the article of Koike and Ochiai in this volume, their result is the consequence of the discussion motivated by the RIMS workshop.

1 Realization as an algebraic surface

We fix an abstract lattice L and its P -part $D_4^3 \oplus \langle -2 \rangle \oplus \langle 2 \rangle$. A $K3$ surface S is called of exact type P when it satisfies $P \cong \text{Pic}(S)$ via an isomorphism $\varphi : H_2(S, \mathbb{Z}) \rightarrow L$. That is a general member of \mathcal{F} . A surface S of exact type P is realized as a double covering surface over $\mathbb{P}^1 \times \mathbb{P}^1$ branching along four bidegree $(1, 1)$ curves H_1, H_2, H_3, H_4 satisfying

$J_1) H_k$ ($k = 1, 2, 3, 4$) is irreducible,

J_2) $H_k \cap H_\ell$ consists of two different points,

J_3) for any different three indices i, j, k we have $H_i \cap H_j \cap H_k = \emptyset$

under some nef condition stated in Theorem 1. Such a surface is given as the complete nonsingular model of the affine variety

$$S = S(x) : w^2 = \prod_{k=1}^4 (x_1^{(k)} st + x_2^{(k)} s + x_3^{(k)} t + x_4^{(k)}),$$

where we use the notation

$$x_k = \begin{pmatrix} x_1^{(k)} & x_2^{(k)} \\ x_3^{(k)} & x_4^{(k)} \end{pmatrix} \in M(2, \mathbb{C}).$$

So the curve H_k ($k = 1, 2, 3, 4$) is given by

$$(s, 1) \begin{pmatrix} x_1^{(k)} & x_2^{(k)} \\ x_3^{(k)} & x_4^{(k)} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = 0.$$

By considering the projection to the s -line we find that a general member S of \mathcal{F} is an elliptic fibred surface with 12 singular fibres of type I_2 corresponding to the intersection points $H_i \cap H_j$ ($i \neq j$). Counting the Euler number we know that $S(x)$ is a $K3$ surface. Let E_{ij}^\pm be the exceptional curves obtained by the blow up processes at the intersections $H_i \cap H_j$ ($i \neq j, 1 \leq i, j \leq 4$). Let π be the projection $S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and set

$$G_i = \frac{1}{2}(\pi^* H_i - \sum_{i \neq j} (E_{ij}^+ + E_{ij}^-)) \quad (i = 1, 2, 3, 4)$$

$$F_s = \pi^* \{t = 0\}, F_t = \pi^* \{s = 0\}.$$

Lemma 1.1 *The sublattice in $H_2(S(x), \mathbb{Z})$ generated by $E_{ij}^+, E_{ij}^-, G_i, F_s, F_t$ is isomorphic to P . So $S(x)$ is a $K3$ surface of type P .*

According to the above Lemma we can define the elements $e_{ij}^\pm, g_i, f_s, f_t$ in the abstract lattice L corresponding to the divisors with capital letters in $H^2(S(x), \mathbb{Z})$ via an isomorphism $\varphi : H_2(S(x), \mathbb{Z}) \rightarrow L$.

Theorem 1 *Let S be a $K3$ surface of type P with an isomorphism φ , and suppose that $\varphi^{-1}(f_s)$ and $\varphi^{-1}(f_t)$ are nef divisors. Then S is realized as the form of $S(x)$.*

2 Period map

2.1 Marking of a $K3$ surface of type P

Definition 2.1 *Let S be a $K3$ surface of type P with an embedding $P \rightarrow L$. We call the triple (S, φ, P) a P marking of S if we have*

- (1) $\varphi : H_2(S, \mathbb{Z}) \rightarrow L$ is an isometry of lattices,
- (2) $\varphi^{-1}(f_s), \varphi^{-1}(f_t), \varphi^{-1}(e_{ij}^\pm), \varphi^{-1}(g_i)$ are effective divisors and $\varphi^{-1}(f_s)$ and $\varphi^{-1}(f_t)$ are nef.

Let $O(L)$ be the group of isometries of the lattice L , and set $O(L, P) = \{g \in O(L) : g(x) = x \text{ for } \forall x \in P\}$. Putting $T = P^\perp$ we set $G = O(T)$ and $G(2) = \{g \in G : g \equiv I \pmod{2}\}$, where we defined $O(T)$ as same as $O(L)$.

Proposition 2.1 *We have $G(2) \cong O(L, P)$.*

Definition 2.2 *Let (S, φ, P) and (S', φ', P) be P markings of S and S' , respectively. An isomorphism $\rho : S \rightarrow S'$ is called an isomorphism of these 2 markings when we have $\varphi = \varphi' \circ \rho_*$. We say 2 markings (S, φ, P) and (S', φ', P) are equivalent when we have a $\gamma \in O(L, P)$ such that (S', φ', P) is isomorphic to $(S, \gamma \circ \varphi, P)$.*

Remark 2.1 *Equivalent markings correspond to the same configuration of the branch locus with the order of H_1, \dots, H_4 of the double covering surfaces. And this relation corresponds to the base change of the $K3$ lattice which preserves the polarization P .*

2.2 Parameter space

From the above argument we can take the configuration space for the family of 4 bidegree $(1, 1)$ curves as a parameters space of our family of P marked $K3$ surfaces. The presentation of H_k has an ambiguity of a constant factor. By considering the projective coordinate transformation of s -space and t -space our parameter space is given by

$$X = \left(PGL(2, \mathbb{C}) \setminus \{(x_1, \dots, x_4)\} = (GL(2, \mathbb{C}))^4 / PGL(2, \mathbb{C}) \right) / (\mathbb{C}^*)^4.$$

For further investigation we need a realization of X as a projective variety. That is suggested in the article of Koike and Ochiai on this same volume.

2.3 Period domain

Let $\Gamma_1, \dots, \Gamma_8$ be a fixed basis of T such that we have the intersection form

$$(\Gamma_i \cdot \Gamma_j) = A := U(2) \oplus (-2I_4).$$

For a P marking (S, φ, P) let Ω be the holomorphic 2-form on S that is unique up to a constant factor. We define the period of (S, φ, P) by

$$\eta = \left[\int_{\varphi^{-1}(\Gamma_1)} \Omega, \dots, \int_{\varphi^{-1}(\Gamma_8)} \Omega \right] \in \mathbb{P}^7.$$

The image of the period mapping for the family of P marked $K3$ surfaces is open dense in the 6-dimensional domain given by

$$D^+ = \{\eta = [\eta_1, \dots, \eta_8] \in \mathbb{P}^7 : {}^t\eta A \eta = 0, {}^t\bar{\eta} A \eta > 0, \Im(\eta_3/\eta_1) > 0\}.$$

We get this fact by using the Riemann-Hodge relation of the period and the Torelli theorem for $K3$ surfaces. It is a bounded symmetric domain of type IV. We set

$$G^+ = \{g \in G : g(D^+) = D^+\}, \quad G(2)^+ = \{g \in G(2) : g(D^+) = D^+\}.$$

We can determine the modular group for the equivalence classes of the P marked surfaces. Namely

Theorem 2 *Let (S, φ, P) and (S', φ', P) be P markings of $K3$ surfaces of type P . Let η and η' be the corresponding periods, respectively. Then these two markings are equivalent if and only if*

$$g(\eta) = \eta'$$

for some $g \in G(2)^+$

Theorem 3 *The modular group $G(2)^+$ is a reflection group.*

Here a transformation

$$R_v : \lambda \mapsto v - 2({}^t v A \lambda / {}^t v A v) v, \quad \lambda \in D^+, v \in \mathbb{Z}^8$$

is called a reflection with the root vector v .

2.4 Degenerate locus

3 Differential equation

We can determine the system of differential equations for the period with 16 variables x_{ij}^k . It becomes a holonomic system of rank 8. So our periods $\int_{\varphi^{-1}(\Gamma_i)} \Omega$ ($i = 1, \dots, 8$) make a basis of the space of solutions for this system defined on a domain $X' = X - V$, where V is the degenerating locus corresponding to the set of $K3$ surfaces of type P which violate some condition of J_1, J_2, J_3 .

So we can consider the monodromy group \mathcal{M} for this system.

Proposition 3.1 *We have $G(2)^+ \subset \mathcal{M}$.*

Remark 3.1 *We have possibly $G(2)^+ = \mathcal{M}$. But we cannot decide it at present, because Some monodromy transformation may cause an interchange of E_{ij}^\pm .*

Remark 3.2 *The image of the degenerating locus V by the period map is consists of 4, 6, 16 hyperplanes (so in total 26 hyperplanes) in the period domain D^+ which correspond to the violation of the condition J_1, J_2, J_3 , respectively.*

4 Transfer of the period domain

The type II domain \mathbf{H} is defined by

$$\mathbf{H} = \mathbf{H}_{II} = \{Z \in M(4, \mathbb{C}) : J_4 Z = {}^t Z J_4, \quad \frac{1}{i}(Z - Z^*) > 0\},$$

where we use the notation

$$J_{2n} = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix}$$

and the member $Z \in \mathbf{H}$ is described in the form

$$Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & t & a & c \\ -t & 0 & b & d \end{pmatrix}.$$

We define the mapping $\psi : \mathbf{H} \rightarrow \mathbf{P}^7$ by

$$\zeta = {}^t [z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8] = {}^t [1, -ad + bc - st, a, d, b, c, -s, t].$$

As a direct translation of the above lemma, we obtain :

Proposition 4.1 *The image of ψ is determined in \mathbf{P}^7 by the following three conditions:*

$$(1) \quad {}^t\zeta(U \oplus U \oplus U \oplus U)\zeta = 0.$$

$$(2) \quad \zeta^*(U \oplus U \oplus U \oplus U)\zeta > 0,$$

where $\zeta^* = {}^t\bar{\zeta}$.

$$(3) \quad \Im\left(\frac{z_3}{z_1}\right) > 0.$$

By straight forward calculation we have the following.

Theorem 4 *The image $\psi(\mathbf{H})$ is transformed to the type IV domain D_{IV}^+ by the map :*

$$\eta = P\zeta$$

with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{1-i} & \frac{1}{1-i} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{1+i} & \frac{1}{1+i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-i}{1-i} & \frac{1}{1-i} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-i}{1+i} & \frac{1}{1+i} \end{pmatrix}. \quad (4.1)$$

So the composite mapping $\delta = P \circ \psi$ gives the isomorphism

$$\delta : \mathbf{H} \cong D_{IV}.$$

Remark 4.1 *The analytic equivalence of the domains D_{IV}^+ and \mathbf{H} is well known. But we are wishing to find the transfer which preserves the modular groups each other.*

4.1 the Quaternion half space

Let \mathbf{F} be the Hamilton quaternion \mathbf{R} -algebra generated by $\{e_1, e_2, e_3, e_4\}$ with

$$e_1 = 1, \quad e_2e_3 = e_4, \quad e_i^2 = -1.$$

Then the ring of integers in \mathbf{F} is given by

$$\mathcal{O}(\mathbf{F}) = \mathbf{Z}e_0 + \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3,$$

where $e_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$.

Proposition 4.2 *The mapping*

$$\varphi : M(n, \mathbf{F}) \rightarrow M(2n, \mathbf{C})$$

defined by

$$\varphi\left(\sum_i A_i e_i\right) = \begin{pmatrix} A_1 e_1 + A_2 e_2 & A_3 e_1 + A_4 e_2 \\ -A_3 e_1 + A_4 e_2 & A_1 e_1 - A_2 e_2 \end{pmatrix}$$

is an injective homomorphism of \mathbf{R} -algebra.

Definition 4.1 We set

$$\text{Sym}(2, \mathbf{F}) = \{X \in M(2, \mathbf{F}) : {}^t \bar{X} = X\},$$

$$\text{Pos}(2, \mathbf{F}) = \{X \in \text{Sym}(2, \mathbf{F}) : X > 0\}.$$

Note that we have

$$X \in \text{Sym}(2, \mathbf{F}) \iff \varphi(X) \in \text{Sym}(4, \mathbf{C}),$$

$$X \in \text{Pos}(2, \mathbf{F}) \iff \varphi(X) \in \text{Pos}(4, \mathbf{C})$$

and it holds also

$$X \in \text{Pos}(2, \mathbf{F}) \iff X = g^* g, \quad \exists g \in \text{GL}(2, \mathbf{F}).$$

Definition 4.2 The quaternion half space is defined by

$$\mathbf{H}(n, \mathbf{F}) = \{X + \sqrt{-1}Y : X \in \text{Sym}(n, \mathbf{F}), Y \in \text{Pos}(n, \mathbf{F})\}.$$

Remark 4.2 (1) We can define the half spaces using \mathbf{R} and \mathbf{C} instead of \mathbf{F} . The half space $\mathbf{H}(n, \mathbf{R})$ is a Siegel half space, and $\mathbf{H}(n, \mathbf{C})$ is the bounded symmetric space of type I.

(2) Two spaces $H(2, \mathbf{F})$ and $\varphi(H(2, \mathbf{F})) \subset H(4, \mathbf{C})$ are isomorphic as complex manifolds via the correspondence φ .

Proposition 4.3 For an element $Z \in \mathbf{H}$ we have a decomposition $Z = X + \sqrt{-1}Y$ with

$$X = \frac{1}{2}(Z + {}^t \bar{Z}), Y = \frac{1}{2\sqrt{-1}}(Z - {}^t \bar{Z}),$$

and so \mathbf{H} is naturally embedded in the half space $H(4, \mathbf{C})$.

Remark 4.3 We can examine the equality $\mathbf{H} = \varphi(H(2, \mathbf{F}))$ by direct calculation.

Set

$$\text{Sp}(2n, \mathbf{F}) = \{g \in \text{GL}(2n, \mathbf{F}) : g^* J_{2n} g = J_{2n}\}$$

and we define $\text{Sp}(2n, \mathbf{C})$ by putting \mathbf{C} instead of \mathbf{F} . The following is wellknown:

Proposition 4.4 The group $\text{Sp}(4, \mathbf{F})$ is generated by

$$J_4, \begin{pmatrix} {}^t \bar{W} & O \\ O & W^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix}, \quad \text{where } W \in \text{GL}(2, \mathbf{F}) \text{ and } S^* = S.$$

So we obtain :

Proposition 4.5

$$\text{Image}(\Phi) = \langle J_8, \begin{pmatrix} A & O \\ O & {}^t \bar{A}^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix} \rangle$$

with

$$A = \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix}, S^* = S.$$

Proposition 4.6 ([4] p.55)

The group $\text{Sp}(4, \mathbf{F}) \cap \text{GL}(4, \mathcal{O}(\mathbf{F}))$ is generated by

$$J_4, \begin{pmatrix} I & S \\ O & I \end{pmatrix}, \begin{pmatrix} U & O \\ O & (U^*)^{-1} \end{pmatrix}$$

where $S \in \text{Sym}(2, \mathcal{O}(\mathbf{F}))$, $U \in \text{GL}(2, \mathcal{O}(\mathbf{F}))$.

Proposition 4.7 The group $\text{Sp}(2n, \mathbf{F})$ is a subgroup of $\text{Aut}(H(n, \mathbf{F}))$ via the action

$$Z \mapsto (AZ + B)(CD + D)^{-1}$$

for an element

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbf{F}).$$

Proposition 4.8 ([4] p.50)

We have

$$\text{Aut}(H(2, \mathbf{F})) = \text{Sp}(4, \mathbf{F}) \cdot \langle \Pi \rangle,$$

and we have

$$\text{Aut}(H(n, \mathbf{F})) = \text{Sp}(2, \mathbf{F})$$

for $n \geq 3$. Where Π indicates the transposition as an element of $M(2, \mathbf{F})$, and \cdot means the semi direct product.

Remark 4.4 The transposition Π acts as an automorphism of $H(n, \mathbf{F})$ only for $n = 2$. If we have $n \geq 3$, it does not preserve the positivity condition $-\sqrt{-1}(Z - Z^*) > 0$.

4.2 Relation between $G^+(\mathbf{Z})$ and $\Gamma(\mathbf{H})$

Definition 4.3 Set

$$H = \begin{pmatrix} O & iE_4 \\ -iE_4 & O \end{pmatrix}, \quad S = \begin{pmatrix} O & J_4 \\ -J_4 & O \end{pmatrix}, \quad L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}$$

$$\text{SO}^*(8, \mathbf{C}) = \{g \in \text{GL}(8, \mathbf{C}) : g^* H g = H, \quad {}^t g S g = S\}.$$

And set

$$\Gamma(\mathbf{H}) = \text{SO}^*(8, \mathbf{Z}[i]) \cdot \langle \iota \rangle,$$

where ι indicates the involution

$$Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & t & a & c \\ -t & 0 & b & d \end{pmatrix} \mapsto Z' = \begin{pmatrix} a & b & 0 & t \\ c & d & -t & 0 \\ 0 & s & a & c \\ -s & 0 & b & d \end{pmatrix}.$$

We can easily examine that $\text{SO}^*(8)$ is a subgroup of $\text{Sp}(8, \mathbf{C})$. We obtain the following by checking the conditions for $\text{SO}^*(8)$.

Proposition 4.9 We have a injective homomorphism of \mathbf{R} -algebra $\Phi : \text{Sp}(4, \mathbf{F}) \rightarrow \text{Sp}(8, \mathbf{C})$ by putting

$$\Phi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{pmatrix},$$

and the image is contained in $\text{SO}^*(8)$.

Remark 4.5 (1) Let \mathcal{O}' denote the order $\mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$ of \mathbf{F} . Then we have

$$\mathrm{SO}^*(8, \mathbf{Z}[i]) \cong \mathrm{Sp}(4, \mathcal{O}')$$

via the mapping φ .

(2) We expect that the isomorphism δ induces injective isomorphisms

$$(\delta^{-1})^* : \Gamma(\mathbf{H}) \rightarrow G^+$$

and

$$\delta^*(G^+(2)) \subset \Gamma(\mathbf{H}).$$

But to get them, it is necessary to proceed more detailed argument on the discrete groups on D^+ and \mathbf{H} . We don't have these results still now.

4.3 Embedding of \mathbf{H} into the Siegel upper space \mathcal{S}_8

We use the following notation:

$$\begin{aligned} \mathcal{S}_8 &= \{\Omega \in GL(8, \mathbf{C}) : {}^t\Omega = \Omega, \Im(\Omega) > 0\}, \\ K &= \begin{pmatrix} O & J_4 \\ J_4 & O \end{pmatrix}, L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix} \\ \mathcal{S}_8(q) &= \{\Omega \in \mathcal{S}_8 : \Omega J_8 = J_8 \Omega, \Omega K = K \Omega, \Omega L = L \Omega\}. \end{aligned}$$

Note that $\{I_8, J_8, K, L\}$ make the basis of the Hamilton quaternionic field.

Proposition 4.10 The domain \mathbf{H}_{II} is embedded in \mathcal{S}_8 by the mapping

$$\rho : Z \mapsto \frac{1}{2} \begin{pmatrix} Z + {}^t Z & i(Z - {}^t Z) \\ -i(Z - {}^t Z) & Z + {}^t Z \end{pmatrix}$$

It induces the biholomorphic equivalence between \mathbf{H} and $\mathcal{S}_8(q)$.

The group $\mathrm{SO}^*(8)$ has an injective embedding into $\mathrm{Sp}(16, \mathbf{R})$ by the mapping λ :

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \Re(A) & -\Im(A) & \Re(B) & -\Im(B) \\ \Im(A) & \Re(A) & \Im(B) & \Re(B) \\ \Re(C) & -\Im(C) & \Re(D) & -\Im(D) \\ \Im(C) & \Re(C) & \Im(D) & \Re(D) \end{pmatrix}.$$

For an element $g \in \mathrm{SO}^*(8)$ we have $\rho^{-1} \circ g \circ \rho = \lambda(g)$.

Proposition 4.11 Put

$$J_{II} = \begin{pmatrix} O & O & J_4 & O \\ O & O & O & -J_4 \\ -J_4 & O & O & O \\ O & J_4 & O & O \end{pmatrix}, \quad \tilde{J}_{16} = J_8 \oplus J_8.$$

Then it holds

$${}^t\lambda(g)J_{II}\lambda(g) = J_{II}, \quad {}^t\lambda(g)\tilde{J}_{16}\lambda(g) = \tilde{J}_{16}$$

for every $g \in \mathrm{SO}^*(8)$. If we put

$$\mathrm{Sp}(q) = \{\gamma \in \mathrm{Sp}(16, \mathbf{R}) : {}^t \gamma J_{\mathrm{II}} \gamma = J_{\mathrm{II}}, \quad \gamma \tilde{J}_{16} = \tilde{J}_{16} \gamma\}$$

the mapping λ induces the isomorphism

$$\mathrm{SO}^*(8) \cong \mathrm{Sp}(q).$$

Epecially the mapping ρ induces an isomorphism

$$\mathrm{SO}^*(8, \mathbf{Z}[i]) \cong \mathrm{Sp}(q) \cap \mathrm{M}(16, \mathbf{Z}).$$

Let Ω be a point on \mathcal{S}_8 , and set $\Lambda_\Omega = \Lambda = \mathbf{Z}^8 + \mathbf{Z}^8 \Omega$. Let V_Ω denote the abelian variety $\mathbf{C}^8 / \Lambda_\Omega$. So we regard \mathcal{S}_8 as the coarse moduli space of principally polarized abelian varieties V_Ω . Note that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ belong to $\mathrm{Sp}(16, \mathbf{Z})$. We can check that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ are contained in the algebra of endomorphisms of Λ provided $\Omega \in \mathcal{S}_8(q)$. Let $\langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle$ be a \mathbf{Q} -algebra generated by $I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L$. Then we have

$$\mathbf{F}_{\mathbf{Q}} \cong \langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle \subset \mathrm{End}(V_\Omega) \quad \text{for } \Omega \in \mathcal{S}_8(q),$$

where $\mathbf{F}_{\mathbf{Q}}$ indicates the Hamilton quaternion algebra over \mathbf{Q} .

Proposition 4.12 *The space $\mathcal{S}_8(q)$ is the coarse moduli space for the family of 8-dimensional abelian variety V with the property*

$$\mathbf{F}_{\mathbf{Q}} \cong \langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle \subset \mathrm{End}(V_\Omega).$$

In this sense we can call $\mathcal{S}_8(q)$ the Shimura variety for the Hamilton quaternion endomorphism algebra $\mathbf{F}_{\mathbf{Q}}$.

5 Corresponding Kuga-Satake varieties

We use the method developped in [12] and [10]. The detailed calculation and argument are exposed in [2] also.

Let us consider the lattice T defined by the intersection matrix $A = U(2) \oplus U(2) \oplus (-2I_4)$ and $V_k = T \otimes k$ ($k = \mathbf{R}$ or \mathbf{Q}). Let $Q(x)$ denote the quadratic form on T and at the same time on V_k . Let $\mathrm{Tens}(T)$ and $\mathrm{Tens}(V_k)$ be the corresponding tensor algebras. And we let $\mathrm{Tens}^+(T)$ and $\mathrm{Tens}^+(V_k)$ denote the subalgebras composed of the parts with even degree in $\mathrm{Tens}(T)$ and $\mathrm{Tens}(V_k)$, respectively. We consider the two sided ideal I in $\mathrm{Tens}^+(V_k)$ generated by elements $x \otimes x - Q(x)$ for $x \in V_k$, and the ideal $I_{\mathbf{Z}}$ in $\mathrm{Tens}(T)$ is defined by the same manner. The corresponding even Clifford algebra is defined by

$$C^+(V_k, Q) = \mathrm{Tens}^+(V_k) / I.$$

By the same manner, we define

$$C^+(T, Q) = \mathrm{Tens}^+(T) / I_{\mathbf{Z}}.$$

We note that $C^+(V_{\mathbf{R}}, Q)$ is a 128 dimensional real vector space and $C^+(T, Q)$ is a lattice in it. So we obtain a real torus

$$\mathcal{T}_{\mathbf{R}} = C^+(V_{\mathbf{R}}, Q) / C^+(T, Q).$$

Let \mathbf{F} denote the quaternion algebra

$$\mathbf{Q} \oplus \mathbf{Q}i \oplus \mathbf{Q}j \oplus \mathbf{Q}ij$$

with $i^2 = j^2 = -1$. By some routine calculations of the Clifford algebra we obtain the following.

Proposition 5.1 *We have an isomorphism of algebras $C^+(V_{\mathbf{Q}}, \mathbf{Q}) \cong M(4, \mathbf{F}) \oplus M(4, \mathbf{F})$.*

Let a complex vector $\underline{\eta} = (\eta_1, \dots, \eta_8)$ be a representative of a point $\eta = [\eta_1, \dots, \eta_8] \in D^+$. So it has an ambiguity of the multiplication by a non zero complex number. Put $\underline{\eta} = s + it$ ($s, t \in \mathbf{R}^8$). If we impose the condition $(st)^2 = -1$ in $C^+(V_{\mathbf{R}}, \mathbf{Q})$, the representative is uniquely determined up to a multiplication by a complex unit. We denote it by

$$\underline{\eta} = m_1(\eta) + im_2(\eta).$$

Put

$$m(\eta) = m_1(\eta)m_2(\eta).$$

It is uniquely determined by η without any ambiguity. According to the imposed condition, the element $m(\eta) \in C^+(V_{\mathbf{R}}, \mathbf{Q})$ defines a complex structure on $C^+(V_{\mathbf{R}}, \mathbf{Q})$ by the left action. It induces a complex structure on the real torus $\mathcal{T}_{\mathbf{R}}$ also.

Let $\{\varepsilon_1, \dots, \varepsilon_8\}$ be a basis of T with the intersection matrix $U(2) \oplus U(2) \oplus (-2I_4)$, and let $\{e_1, \dots, e_8\}$ be a orthonormal basis of V given by

$$(e_1, \dots, e_8) = (\varepsilon_1, \dots, \varepsilon_8) \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \oplus (I_4) \right).$$

Then the corresponding intersection matrix takes the form $I_2 \oplus (-I_2) \oplus (-2I_4)$.

Let ι be an involution on $C^+(V, \mathbf{Q})$ induced from the transformation

$$\iota : e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k} \mapsto e_{i_k} \otimes \dots \otimes e_{i_2} \otimes e_{i_1}$$

for the basis. Set $\alpha = 4e_2e_1$. According to the method in [St] we know that

$$E(x, y) = \text{tr}(\alpha x'y)$$

determines a Riemann form. So the triple $(\mathcal{T}_{\mathbf{R}}, m(\eta), E(x, y))$ determines an abelian variety. We denote it by $A^+(\eta)$, that is so called the Kuga-Satake variety attached to the $K3$ surface corresponding to the period η . In this way we can construct a family of abelian varieties

$$\mathcal{A}^+ = \{A^+(\eta) : \eta \in D^+\}$$

induced from the lattice T parameterized by the domain D^+ . We can construct the "conjugate family"

$$\mathcal{A}^- = \{A^-(\eta) : \eta \in D^-\}$$

parameterized by

$$D^- = \{\eta = [\eta_1, \dots, \eta_8] : {}^t\eta A \eta = 0, {}^t\bar{\eta} A \eta > 0, \Im(\eta_3/\eta_1) < 0\}$$

by the same procedure with the Riemann form $E^-(x, y) = -\text{tr}(\alpha x'y)$. The right action of $C^+(V_{\mathbf{Q}}, \mathbf{Q})$ on $C^+(V_{\mathbf{R}}, \mathbf{Q})$ commutes with the left action of $m(\eta)$. So we have

$$C^+(T_{\mathbf{Q}}) \subset \text{End}(A^{\pm}(\eta)) \otimes \mathbf{Q}$$

for any $A^{\pm}(\eta)$. For a general member $\eta \in D^+$, the endomorphism ring is given by

$$\text{End}_{\mathbf{Q}}(A(\eta)) = \text{End}(A(\eta)) \otimes \mathbf{Q} \cong C^+(V_{\mathbf{Q}}).$$

According to Proposition 6.1 we obtain :

Theorem 5 For a general member $\eta \in \mathcal{D}^+$, $A^+(\eta)$ is isogenous to a product of abelian varieties $(A_1(\eta) \times A_2(\eta))^4$ where $A_1(\eta)$ and $A_2(\eta)$ are 8-dimensional simple abelian varieties with $\text{End}_{\mathbf{Q}}(A_i(\eta)) = \mathbf{F}_{\mathbf{Q}}$ ($i = 1, 2$).

Remark 5.1 Here we describe the relation between $A_1(\eta)$ and $A_2(\eta)$. Now we define the linear involution $*$ on $V_{\mathbf{R}}$ by

$$e_1^* = -e_1 \quad \text{and} \quad e_i^* = e_i \quad (i = 2, \dots, 8).$$

It can be extended on $C^+(V_{\mathbf{R}}, \mathbf{Q})$ as an automorphism of algebra. We define an involution σ on \mathcal{D} :

$$\sigma : \mathcal{D} \longrightarrow \mathcal{D}, \quad (\eta_{\infty}, \dots, \eta_{\mathbf{N}}) \mapsto (-\eta_{\mathbf{E}}, -\eta_{\infty}, \eta_{\mathbf{D}}, \dots, \eta_{\mathbf{N}}).$$

So we have $\mathcal{D}_+^{\sigma} = \mathcal{D}_-$. It is easy to check that we have

$$A_2(\eta) \sim A_1(\eta^{\sigma}), \quad A_1(\eta) \sim A_2(\eta^{\sigma}),$$

where \sim indicates the isogenous relation.

E. Freitag and C. F. Hermann [1] study a similar family of lattice K3 surfaces from a different view point. We think that it should be clarified the exact relation between their family and our \mathcal{F} .

References

- [1] E. Freitag and C. F. Hermann, Some modular varieties of low dimension, preprint.
- [2] K. Koike, The Kuga - Satake variety attached to the double covering of $\mathbf{P}^1 \times \mathbf{P}^1$ branched along four curves of bidegree (1, 1), preprint.
- [3] K. Koike, H. Shiga, N. Takayama and T. Tsutsui, Study on the family of K3 surfaces induced from the lattice $(D_4)^3 \oplus \langle -2 \rangle \oplus \langle 2 \rangle$, Int. J. Math., 12(2001), 1049 - 1085.
- [4] A. Krieg, Modular forms on half-spaces of quaternions. Springer LNM 1143(1985).
- [5] M. Kitazume, K. Koike, and H. Shiga, Notes on a certain integral orthogonal group and its quotient by the principal congruence subgroup of level 2, Kyushu J. Math., 54(2000), 257 - 271 .
- [6] K. Matsumoto, Theta functions on the bounded symmetric domain of type $I_{2,2}$ and the period map of a 4 - parameter family of K3 surfaces, Math. Ann., 295(1993), 383 - 409.
- [7] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4 - parameter family of K3 surfaces and the hypergeometric function of type (3,6), Int. J. Math., 3(1992), 1 - 164.
- [8] K. Matsumoto and M. Yoshida, Certain reflection groups acting on symmetric domains of type IV, Mem. Fac. Sci. Kyushu Univ., Ser. A, vol. 46(1992), 115 - 128.
- [9] K. Paranjape, Abelian varieties associated to certain K3 surfaces, Compositio Math., 68(1988), 11 - 22.
- [10] I. Satake, Clifford algebras and families of Abelian varieties, Nagoya Math.J. 27(1966), 435-466.

- [11] H. Shiga, On the arithmetic properties of the modular function induced from the hypergeometric differential equation of type $(3, 6)$, Proc. 7th Int. Coll. Differential Equations Provdiv 1996, 381 - 388.
- [12] B. van Geemen, Kuga - Satake varieties and the Hodge conjecture, To appear in Proc. NATO ASI and CRM Summer School Banff 1998.